Theoretical Explanation of Heat Transfer in Laminar Region of Bingham Fluid

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From time to time, A.I.Ch.E. Journal presents translations of certain technical articles written by our Japanese colleagues in their own language. These translations are made by Kenzi Etani, who received his B.S. in chemical engineering in 1953 at the Tokyo Institute of Technology and his M.S. in 1955 at M.I.T. He is associated with Stone & Webster and is an associate member of American Institute of Chemical Engineers. He is also a member of the Society of Chemical Engineers, Japan, and the Japan Oil Chemists' Society. His offer to help break down the language barrier is acknowledged.

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Abstracts, notation, literature cited, tables, and figure captions not published here appear in English in the original paper. No figures will be reproduced in these translations.

The purpose of this study is to derive theoretically the average temperature difference in a Bingham fluid. Bingham flow is quite different from viscous flow and consists of two patterns, plug and nonplug flow. Figure 1 shows a schematic diagram.

The theoretical solution of the nonplug flow was computed by employing Graetz's (*2*), Nusselt's (*7*) and the author's (*5*) solutions. Since temperature in plug flow varies along the y axis, Stokes's findings (*6* and *11*) were employed. Calculations were made at relative plug radius = 0.5.

THEORETICAL SOLUTION OF HEAT TRANSFER OF BINGHAM FLOW

Theoretical Solution

In the case of laminar flow in the pipes the basic differential equation for heat transfer is expressed by

$$u_y \frac{\partial t}{\partial y} = k\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}\right) \quad (1.1)$$

where

$$k \equiv \lambda / C_p \rho \quad (1.2)$$

It is assumed that physical constants, such as specific heat and thermal conductivity, are constant and that wall temperature t_w is constant regardless of position along the y axis.

In well-developed Bingham flow the velocity distribution is expressed by Oyama and Ito (*8*) as follows: In nonplug flow

$$u_y = \frac{(1-a)^2}{2a\alpha}\bar{u}\left\{1 - \left(\frac{r - r_p}{R - r_p}\right)^2\right\} \quad (1.3)$$

In plug part

$$u_{ap} = (1-a)^2 \bar{u}/2a\alpha \quad (1.4)$$

Velocity distribution and plug radius are assumed to be constant with temperature changes.

In nonplug flow θ is assumed to be

$$\theta = (t - t_w)/(t_1 - t_w) \quad (1.5)$$

From Equations (1.1), (1.3), and (1.5)

$$\frac{(1-a)^2}{2a\alpha}\bar{u}\left\{1 - \left(\frac{r - r_p}{R - r_p}\right)^2\right\}\frac{\partial \theta}{\partial y} = k\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r}\right) \quad (1.6)$$

Boundary conditions are

$$\begin{aligned} \theta &= 1, (t = t_1) \quad \text{at} \quad y = 0 \\ \theta &= 0, (t = t_w) \quad \text{at} \quad r = R \end{aligned} \quad (1.7)$$

In Equation (1.6)

$$S = 2a\alpha k/(1-a)^2 \bar{u} \quad (1.8)$$

The particular solution of Equation (1.6) is obtained by the separation of variables method:

$$\theta = A \exp\left[-Sy/b^2\right] \cdot P \quad (1.9)$$

Accordingly Equation (1.6) becomes:

$$\frac{d^2P}{d\mu^2} + \frac{1}{\mu}\frac{dP}{d\mu} + \left[1 - \left(\frac{\mu - \xi}{\beta - \xi}\right)^2\right]P = 0 \quad (1.10)$$

where

$$\left.\begin{aligned} r/b &= \mu \\ R/b &= \beta \\ r_P/b &= \xi \end{aligned}\right\} \quad (1.11)$$

For the particular solution of Equation (1.10)

$$P = \sum_{m=0}^{\infty} B_m \mu^m \quad (1.12)$$

By substituting Equation (1.12) in Equation (1.10) one can determine B_m as

$$\begin{aligned} B_0 &= 1 \\ B_1 &= 0 \\ B_2 &= \frac{-1}{4}\left\{\frac{(1-a)^2 - a^2}{(1-a)^2}\right\} \\ B_m &= \frac{-1}{m^2}\left[\left\{\frac{(1-a)^2 - a^2}{(1-a)^2}\right\} \cdot B_{m-2} + \frac{2a}{(1-a)^2\beta}B_{m-3} - \frac{1}{(1-a)^2\beta^2}B_{m-4}\right] \end{aligned} \quad (1.13)$$

From Equation (1.11)

$$r_p/R = \xi/\beta = a \quad (1.14)$$

At the pipe wall $r = R$, from Equation (1.1) $\mu = \beta$; from Equation (1.7) the boundary condition is $\theta = 0$. Thus Equation (1.12) must be

$$P(\beta) = \sum_{m=0}^{\infty} B_m \beta^m = 0 \quad (1.15)$$

The values of β_0, $\beta_1 \cdots$ are the roots of Equation $P(\beta) = 0$. Since β_{01}, $\beta_1 \cdots$ are changed with relative plug radius a, the following calculation was based on $a = 0.5$. The solution of Equation (1.6) would be

$$\theta = \sum_{m=0}^{\infty} A_m \exp\left[-\frac{S\beta_m{}^2 y}{R^2}\right] \cdot P_m(r/R) \quad (1.16)$$

P_m can be obtained from the following equation, since one knows that $\mu = \zeta\beta$ from Equation (1.11).

$$P_m = \sum_{m=0}^{\infty} B_m(\beta_m) \times (\zeta\beta_m)^m \quad (1.17)$$

From Equation (1.7) $y = 0$, and $\theta = 1$

$$1 = \sum_{m=0}^{\infty} A_m P_m \quad (1.18)$$

From Equation (1.10)

$$\int_a^1 P_m \zeta\left[1 - \left(\frac{\zeta - a}{1 - a}\right)^2\right] d\zeta = -\frac{1}{\beta_m{}^2}\left[\zeta\frac{dP_m}{d\zeta}\right]_{\zeta=a}^{\zeta=1} \quad (1.19)$$

Therefore from Equations (1.18) and (1.19)

$$A_m = \frac{-\frac{1}{\beta_m^2}\left[\zeta \frac{dP_m}{d\zeta}\right]_{\zeta=a}^{\zeta=1}}{\int_a^1 P_m^2 \zeta\left[1-\left(\frac{\zeta-a}{1-a}\right)^2\right] d\zeta} \tag{1.20}$$

A_m can be defined from the preceding equation. If temperature t_{rp} at r_p is expressed as θ_{rp}

$$\theta_{rp} = \sum_{m=0}^{\infty} A_m e^{-(s\beta_m^2/R^2)y} P_m(a) \tag{1.21}$$

In the case of plug flow, from Equations (1.1) and (1.4),

$$\frac{(1-a)^2}{2a\alpha}\bar{u}\frac{\partial t_p}{\partial y} = k\left(\frac{\partial^2 t_p}{\partial y^2} + \frac{1}{r}\frac{\partial t_p}{\partial r}\right) \tag{1.22}$$

Since velocity distribution is independent of radius, the solution of Equation (1.22) is expressed as a Bessel function. The temperature at $r = r_p$ is not constant but varies along the y axis as shown in Equation (1.21). Equation (1.21) can be solved by Stokes's method (*6* and *11*). If θ_p is expressed as

$$\theta_p = (t_p - t_w)/(t_1 - t_w) \tag{1.23}$$

Equation (1.22) is expressed as

$$\frac{\partial \theta_p}{\partial y} = S\left(\frac{\partial^2 \theta_p}{\partial y^2} + \frac{1}{r}\frac{\partial \theta_p}{\partial r}\right) \tag{1.24}$$

Boundary conditions are

$$\theta_p = 1, (t_p = t_1) \quad \text{at} \quad y = 0$$

$$\theta_p = F(y) = \sum_{m=0}^{\infty} A_m e^{-(s\beta_m^2/R^2)y} P_m(a) \quad \text{at} \quad r = r_p \tag{1.25}$$

$$(\partial\theta/\partial r)_{r=r_p} = (\partial\theta_p/\partial r)_{r=r_p} \quad \text{at} \quad r = r_p$$

if

$$\theta_p = \frac{2}{r_p^2}\sum_{s=1}^{\infty} \frac{Y_s J_0(\psi_s \cdot r/r_p)}{[J_1(\psi_s)]^2} \tag{1.26}$$

where

$$Y_s = \int_0^{r_p} \theta_p(r, y) J_0 \cdot \left(\psi_s \frac{r}{r_p}\right) r\, dr \tag{1.27}$$

ψ_s is the sth root of the Bessel function $J_0(\psi) = 0$. From Equations (1.24) and (1.25)

$$\frac{\partial^2 \theta_p}{\partial r^2} + \frac{1}{r}\frac{\partial \theta_p}{\partial r} = \frac{2}{r_p^2}\sum_{s=1}^{\infty}\frac{J_0(\psi_s \cdot r/r_p)}{[J_1(\psi_s)]^2} \cdot \int_0^{r_p} J_0\left(\psi_s \frac{r}{r_p}\right)\frac{\partial}{\partial r} \cdot \left\{r\frac{\partial \theta_p(r, y)}{\partial r}\right\} dr = \frac{2}{r_p^2}\sum_{s=1}^{\infty}\frac{J_0(\psi_s \cdot r/r_p)}{[J_1(\psi_s)]^2} \cdot \left[\psi_s F(y) J_1(\psi_s) - \frac{\psi_s^2}{r_p^2} Y_s\right] \tag{1.28}$$

Also

$$\frac{\partial \theta_p}{\partial y} = \frac{2}{r_p^2}\sum_{s=1}^{\infty}\frac{J_0(\psi_s \cdot r/r_p)}{[J_1(\psi_s)]^2}\frac{dY_s}{dy} \tag{1.29}$$

From Equations (1.24), (1.28), and (1.29),

$$\frac{dY_s}{dy} + \frac{\psi_s^2}{r_p^2} S Y_s = S\psi_s F(y) J_1(\psi_s) \tag{1.30}$$

This equation can be solved easily. The result is

$$Y_s = S\psi_s J_1(\psi_s) \cdot \int_0^y F(v) e^{-(s\psi_s^2/r_p^2)(y-v)} \cdot dv + Ce^{-(s\psi_s^2/r_p^2)y} \tag{1.31}$$

where C in Equation (1.31) is a constant of integration and can be defined when Y_s is fixed at $y = 0$. From boundary conditions $y = 0$ and $\theta_p = 1$ Equation (1.26) must be

$$(Y_s)_{y=0} = \frac{r_p^2}{\psi_s} J_1(\psi_s) \tag{1.32}$$

Therefore Equation (1.31) is

$$Y_s = S\psi_s J_1(\psi_s) \cdot \int_0^y F(v) e^{-(s\psi_s^2/r_p^2)(y-v)}\, dv + \frac{r_p^2}{\psi_s} J_1(\psi_s) e^{-(s\psi_s^2/r_p^2)y} \tag{1.33}$$

From these equations Equation (1.26) becomes

$$\theta_p = 2\sum_{s=1}^{\infty}\frac{J_0(\psi_s \cdot r/r_p)}{\psi_s J_1(\psi_s)} e^{-(S\psi_s^2/r_p^2)y} + \frac{2S}{r_p^2}\sum_{s=1}^{\infty}\frac{\psi_s J_0(\psi_s \cdot r/r_p)}{J_1(\psi_s)} \times \int_0^y F(v) e^{-(s\psi_s^2/r_p^2)(y-v)}\, dv \tag{1.34}$$

From boundary conditions (1.25)

$$F(v) = \sum_{m=0}^{\infty} A_m e^{-(s\beta_m^2/R^2)v} P_m(a) \tag{1.35}$$

Substituting Equation (1.35) in Equation (1.34) one can obtain

$$\theta_p = 2\sum_{s=1}^{\infty}\frac{J_0(\psi_s \cdot r/r_p)}{\psi_s J_1(\psi_s)} e^{-(s\psi_s^2/r_p^2)y} + \sum_{m=0}^{\infty} A_m e^{-(s\beta_m^2/R^2)y} P_m(a) + 2\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\frac{A_m P_m(a) a^2\beta_m^2 J_0(\psi_s \cdot r/r_p)}{(\psi_s^2 - a^2\beta_m^2)\psi_s J_1(\psi_s)} e^{-(s\beta_m^2/R^2)y} - 2\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\frac{A_m P_m(a)\psi_s J_0(\psi_s \cdot r/r_p)}{(\psi_s^2 - a^2\beta_m^2) J_1(\psi_s)} e^{-(s\psi_s^2/r_p^2)y} \tag{1.36}$$

The temperature at any radius in the plug flow can be obtained from this equation.

From the last condition in Equation (1.25)

$$\sum_{m=0}^{\infty} A_m e^{-(s\beta_m^2/R^2)y}\left(\zeta\frac{\partial P_m}{\partial \zeta}\right)_{\zeta=a} = -2\sum_{s=1}^{\infty} e^{-(s\psi_s^2/r_p^2)y} - 2\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\frac{A_m P_m(a) a^2\beta_m^2}{\psi_s^2 - a^2\beta_m^2} e^{-(s\beta_m^2/R^2)y} + 2\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\frac{A_m P_m(a)\psi_s^2}{\psi_s^2 - a^2\beta_m^2} e^{-(s\psi_s^2/r_p^2)y} \tag{1.37}$$

To obtain the heat transfer coefficient the average temperature should be known. If the average temperature t_M at $y = y$ is expressed as θ_M,

$$\theta_M = (t_M - t_w)/(t_1 - t_w) \tag{1.38}$$

The small average temperature is

$$\theta_M = \frac{\int_0^{r_p} 2\pi r u_{yp}\theta_p\,dr + \int_{r_p}^{R} 2\pi r u_y \theta\,dr}{\pi R^2 \bar{u}}$$

$$= -2\frac{(1-a)^2}{2a\alpha}\sum_{m=0}^{\infty} A_m e^{-(s\beta_m^2/R^2)y}\frac{1}{\beta_m^2}\left[\zeta\frac{dP_m}{d\zeta}\right]_{\zeta=a}^{\zeta=1}$$

$$+ 4a^2\frac{(1-a)^2}{2a\alpha}\left[\sum_{s=1}^{\infty}\frac{e^{-(s\psi_s^2/r_p^2)y}}{\psi_s^2}\right. \tag{1.39}$$

$$+\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\frac{A_mP_m(a)}{\psi_s^2 - a^2\beta_m^2}e^{-(s\beta_m^2/R^2)y}$$

$$\left.-\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\frac{A_mP_m(a)}{\psi_s^2 - a^2\beta_m^2}e^{-(s\psi_s^2/r_p^2)y}\right]$$

The relationship between the local heat transfer rate Q_y at $y = y$ and the local coefficient of heat transfer h_y is expressed as

$$Q_y = -\lambda(t_1 - t_w)\left(\frac{\partial\theta}{\partial\gamma}\right)_{r=R} \tag{1.40}$$

$$= (t_1 - t_w)hy\theta_M$$

Therefore h_y is expressed as

$$h_y = \frac{-\frac{\lambda}{R}\sum_{m=0}^{\infty} A_m e^{-(s\beta_m^2/R^2)y}\left(\zeta\frac{\partial P_m}{\partial\zeta}\right)_{\zeta=1}}{\theta_M} \tag{1.41}$$

If the same solution as in laminar flow of pseudoplastic fluid (*5*) is applied, the average coefficient of heat transfer h_M is expressed as

$$h_M = -\frac{2\lambda}{Dl}\int_0^l \frac{\sum_{m=0}^{\infty} A_m e^{-(s\beta_m^2/R^2)y}\left(\zeta\frac{\partial P_m}{\partial\zeta}\right)_{\zeta=1}}{\theta_M}dy \tag{1.42}$$

In Equation (1.39) θ_M is a function of y, and P_m is a function of ζ independent of y; that is $(\partial P_m/\partial\zeta = dP_m/\partial\zeta)$. From Equation (1.37) the following equation can be obtained:

$$\frac{d\theta_M}{dy} = 2\frac{k}{\bar{u}R^2}\sum_{m=0}^{\infty} A_m e^{-(s\beta_m^2/R^2)y}\cdot\left(\zeta\frac{dP_m}{d\zeta}\right)_{\zeta=1}$$

Therefore Equation (1.42) becomes

$$\frac{h_MD}{\lambda} = \frac{1}{4}\left(\frac{\bar{u}D}{kl}\right)\ln\left(\frac{1}{\theta_{Ml}}\right)$$

$$= \frac{1}{\pi}\left(\frac{WC_p}{\lambda l}\right)\ln\left(\frac{1}{\theta_{Ml}}\right) \tag{1.43}$$

If at $y = 1$, the average temperatures t_M and θ_M in Equation (1.38) are expressed as t_{Ml} and θ_{Ml}, respectively,

$$\theta_{Ml} = \frac{-(1-a)^2}{2a\alpha}\cdot\sum_{m=0}^{\infty} A_m e^{-(s\beta_m^2/R^2)l}\cdot\frac{1}{\beta_m^2}\left[\zeta\frac{dP_m}{d\zeta}\right]_{\zeta=a}^{\zeta=1}$$

$$+ 4a^2\frac{(1-a)^2}{2a\alpha} \tag{1.44}$$

$$\left[\sum_{s=1}^{\infty}\frac{e - (s\psi_s^2/r_p^2)l}{\psi_s^2}\right.$$

$$+\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\frac{A_mP_m(a)}{\psi_s^2 - a^2\beta_m^2}\cdot e^{-(s\beta_m^2/R^2)l}$$

$$\left.-\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\frac{A_mP_m(a)}{\psi_s^2 - a^2\beta_m^2}\cdot e^{-(s\psi_s^2/r_p^2)l}\right]$$

The average-temperature difference is expressed as a logarithmic average by heat balance around the pipe.

In these theoretical calculations more calculations of β_m would result in greater application of the theoretical solution. To compute β_m to such an extent however requires much labor. It is convenient to use the following approximate solutions when the Graetz number is large.

Approximate Solution

The basic equation for heat transfer is presented as

$$u_y\frac{\partial t}{\partial y} = k\frac{\partial^2 t}{\partial x^2} \tag{2.1}$$

Velocity u_y is the same as that in Equation (1.3) and is near the pipe wall. If $r = R - x$ and $x \ll R$, the following equation is derived:

$$u_y = \frac{4(1-a)}{4a\alpha}\frac{\bar{u}x}{R} \tag{2.2}$$

Equation (2.1) becomes

$$\frac{4(1-a)\bar{u}}{4a\alpha kR}x\frac{\partial t}{\partial y} = \frac{\partial^2 t}{\partial x^2} \tag{2.3}$$

If the following equation is defined as

$$\frac{4(1-a)\bar{u}}{4a\alpha kR} = U \tag{2.4}$$

Equation (2.3) can be solved:

$$\left.\begin{aligned}\frac{h_MD}{\lambda} &= 1.62\left\{\left(\frac{\bar{u}D^2}{kl}\right)\left(\frac{1-a}{4a\alpha}\right)\right\}^{\frac{1}{3}}\\ &= 1.75\left\{\left(\frac{WC_p}{\lambda l}\right)\left(\frac{1-a}{4a\alpha}\right)\right\}^{\frac{1}{3}}\end{aligned}\right\} \tag{2.5}$$

$$\theta_{Ml} = 1 - 5.50\left(\frac{WC_p}{\lambda l}\right)^{-\frac{2}{3}}\cdot\left(\frac{1-a}{4a\alpha}\right)^{\frac{1}{3}} \tag{2.6}$$

and Equation (2.5) is applicable in the following range:

$$\left(\frac{WC_p}{\lambda l}\right)\left(\frac{1-a}{4a\alpha}\right) > 1{,}000 \tag{2.7}$$

In other words when the left term in the equation is greater than 1,000, an approximate solution can be used. $\{(1-a)/4a\alpha\}$ is the only function of a. Calculated results are shown in Figure 2 and Table 1.

Theoretical Solution When Flow Is Rodlike and Wall Temperature Is Not Constant

These calculations were based on constant wall temperature. In many practical cases however wall temperature is not constant is a function of y position. Moreover the velocity distribution generally is a function of ζ; therefore computation requires much labor. The following calculations are based on rod-like flow $a = 1$, where wall temperature is not constant.

When $a = 1$, Equation (1.8) becomes

$$S = k/\bar{u} \tag{3.1}$$

If $S = k/\bar{u}$ and $r_p = R$ are substituted in Equation (1.34), the wall temperature is the solution of $F(y)$.

$$\theta_p = 2\sum_{s=1}^{\infty}\frac{J_0\left(\psi_s\frac{r}{R}\right)}{\psi_sJ_1(\psi_s)}$$

$$\cdot e^{-\pi\psi_s^2(\lambda y/WC_p)}$$

$$+ 2\pi \frac{\lambda}{WC_p} \sum_{s=1}^{\infty} \frac{\psi_s J_0\left(\psi_s \frac{r}{R}\right)}{J_1(\psi_s)} \cdot \int_0^y F(v) e^{-\pi\psi_s^2(\lambda/WC_p)(y-v)}\, dv \quad (3.2)$$

where θ_p is the same as that in Equation (1.23). Since wall temperature varies along the y axis, t_w is defined as the wall temperature at $y = 0$, and t_{wy} as the wall temperature at $y = y$.

$$t_{wy} = (1 + C_y)t_w \quad (3.3)$$

Wall temperature is assumed to change linearly. The boundary condition in this case is

$$\theta_p = 1, (t = t_1) \quad \text{at} \quad y = 0$$

$$\theta_p = F(y) = Ct_w y/(t_1 - t_w) \quad \text{at} \quad r = R \quad (3.4)$$

From Equations (3.2) and (3.4)

$$\theta_p = 2 \sum_{s=1}^{\infty} \frac{J_0\left(\psi_s \frac{r}{R}\right)}{\psi_s J_1(\psi_s)} \cdot e^{-\pi\psi_s^2(\lambda y/WC_p)} + \frac{Ct_w}{t_1 - t_w} \frac{y}{\pi(\lambda y/WC_p)} \cdot \left[\pi\left(\frac{\lambda y}{WC_p}\right) - \frac{1}{4}\left\{1 - \left(\frac{r}{R}\right)^2\right\} + 2 \sum_{s=1}^{\infty} \frac{J_0(\psi_s \cdot r/R)}{\psi_s^3 J_1(\psi_s)} \cdot e^{-\pi\psi_s^2(\lambda y/WC_p)}\right] \quad (3.5)$$

This equation is the same as that derived from integration (*1*). When Equation (3.5) was derived, the Fourier-Bessel series (*3*) was used.

$$\sum_{s=1}^{\infty} \frac{J_0(\psi_s \cdot r/R)}{\psi_s^3 J(\psi_s)} = \frac{1}{8}\left\{1 - \left(\frac{r}{R}\right)^2\right\}$$

Average temperature is expressed as;

$$\theta_M = 4 \sum_{s=1}^{\infty} \frac{1}{\psi_s^2} e^{-\pi\psi_s^2(\lambda y/WC_p)} + \frac{ct_w}{t_1 - t_w} \frac{y}{\pi(\lambda y/WC_p)} \cdot \left[\pi\left(\frac{\lambda y}{WC_p}\right) - \frac{1}{8} + 4 \cdot \sum_{s=1}^{\infty} \frac{1}{\psi_s^4} e^{-\pi\psi_s^2(\lambda y/WC_p)}\right] \quad (3.6)$$

The local coefficient of heat transfer and the average coefficient of heat transfer are considered to be the same as when the theoretical solution was obtained.

$$\frac{h_M D}{\lambda} = \frac{1}{\pi}\left(\frac{WC_p}{\lambda l}\right) \ln\left(\frac{1}{\theta_{Ml}}\right) \quad (3.7)$$

The average-temperature difference is expressed as a logarithmic mean. If the average temperature θ_M in Equation (3.6) is expressed as θ_{Ml} at $y = 1$,

$$\theta_{Ml} = 4 \sum_{s=1}^{\infty} \frac{1}{\psi_s^2} e^{-\pi\psi_s^2(\lambda l/WC_p)} + \frac{Ct_w}{t_1 - t_w} \frac{l}{\pi(\lambda l/WC_p)} \cdot \left[\pi\left(\frac{\lambda l}{WC_p}\right) - \frac{1}{8} + 4 \sum_{s=1}^{\infty} \frac{1}{\psi_s^4} e^{-\pi\psi_s^2(\lambda l/WC_p)}\right] \quad (3.8)$$

When wall temperature is constant, $C = 0$ in Equation (3.3), and average temperature θ_{Ml} is expressed by only the first term in Equation (3.8). The average Nusselt number can be calculated by giving the values $(WC_p/\lambda l)$, t_w, t_1, and C in Equation (3.8).

CALCULATIONS AND CONSIDERATIONS

In the calculation of $a = 0.5$, since B_m is a function of β, β_m' is obtained by calculating $\sum_{m=0}^{\infty} B_m\beta_m$ and is shown in Table 2, where odd numbers of B_m' are all zero. From Equation (1.15)

$$P(\beta) = \sum_{m=0}^{\infty} B_m\beta^m = \sum_{m=0}^{\infty} B_m'\beta_m = 0 \quad (4.1)$$

When one calculates the roots β_m in Equation (4.1)

$$\beta_0 = 2.64150$$
$$\beta_1 = 6.6544 \quad (4.2)$$
$$\beta_2 = 10.654$$

P_m was calculated from Equation (1.17), and the results are shown in Figure 3 and Table 3. The value $1/\beta_m^2 \cdot [\zeta dP_m/d\zeta]_{\zeta=a}^{\zeta-1}$ is given as

$$\frac{1}{\beta_0^2}\left[\zeta \frac{dP_0}{d\zeta}\right]_{\zeta=a}^{\zeta=1} = -9.8878 \times 10^{-2}$$

$$\frac{1}{\beta_1^2}\left[\zeta \frac{dP_1}{d\zeta}\right]_{\zeta=a}^{\zeta=1} = 7.6542 \times 10^{-2} \quad (4.3)$$

$$\frac{1}{\beta_2^2}\left[\zeta \frac{dP_2}{d\zeta}\right]_{\zeta=a}^{\zeta=1} = -2.265 \times 10^{-2}$$

A_m was obtained by graphical integration from Equation (1.20)

$$A_0 = 1.5219$$
$$A_1 = -2.660 \quad (4.4)$$
$$A_2 = 1.504$$

The value of θ_{Ml} was obtained from Equation (1.14):

$$\theta_{Ml} = 0.9192e^{-15.562(\lambda l/WC_p)} - 0.0892e^{-51.746(\lambda l/WC_p)} + 0.502e^{-98.54(\lambda l/WC_p)} - 0.303e^{-252.58(\lambda l/WC_p)} + \cdots \quad (4.5)$$

The relationship between $(1 - \theta_{Ml})$ and the Graetz number must be related to the relative plug radius. Two extreme cases, that is $a = 1$ and $a = 0$, were considered. When $a = 1$, $(1 - \theta_{Ml})$ is obtained by using $C = 0$ (see Section on theoretical solution when flow is rodlike and wall temperature is not constant).

$$1 - \theta_{Ml} = 1 - 4 \cdot \sum_{m=0}^{\infty} e^{-\pi\psi_s^2(\lambda l/WC_p)}/\psi_s^2 \quad (4.6)$$

When the Graetz number in Equation (4.6) is increased, the approximate solution at $n = \infty$ is obtained:

$$1 - \theta_{Ml} = 4(WC_p/\lambda l)^{-1/2} \quad (4.7)$$

When $a = 0$, the fluid corresponds to viscous fluid; that is, $n = 1$. When the Graetz number is above 1,000, the relationship between $(1 - \theta_{Ml})$ and the Graetz number can be obtained by using $a = 0$ in Equation (2.6).

$$1 - \theta_{Ml} = 5.50(WC_p/\lambda l)^{-2/3} \quad (4.8)$$

When $a = 0.5$, Equation (4.5) can be used. When the Graetz number is large, the following equation obtained from Equation (2.6) may be used:

$$1 - \theta_{Ml} = 6.17(WC_p/\lambda l)^{-2/3} \quad (4.9)$$

The relation between $(1 - \theta_{Ml})$ and the Graetz number at three a values is shown in Figure 4.

The average Nusselt number vs. Graetz number calculated from Equation (1.43) is shown in Figure 5; the relation is linear when the Graetz number is greater than 100. By comparing Equation (1.43) with (2.5) the following equation holds when $\{(WC_p/\lambda l)(1 - a)/4a\alpha\} > 100$.

$$\frac{h_M D}{\lambda} = 1.75\left\{\left(\frac{WC_p}{\lambda l}\right)\left(\frac{1 - a}{4a\alpha}\right)\right\}^{\frac{1}{3}} \quad (4.10)$$

(Continued on page 8M)